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Nature of quantum recurrences in coupled higher dimensional systems

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Abstract. We investigate recurrence phenomena in coupled two degrees of freedom systems. It is shown that an initial well localized wave packet displays recurrences even in the presence of coupling in these systems. We discuss the interdependence of time scales namely classical period and quantum revival time and explain the significance of initial conditions.

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Characteristics of quantum systems, which exhibit chaos in classical domain, have posed interesting questions for researchers. In one of these systems, namely, hydrogen atom in microwave field, the discovery of phenomenon of quantum dynamical localization [1] proved as a land mark in the young field of quantum chaos. Fishman et al. connected dynamical localization with Anderson localization [2], and proved it to be a generic property of periodically driven quantum systems. Later, the phenomenon was experimentally observed [3]. In this paper, we develop analytical treatment for quantum recurrence phenomena [4,5] in systems which may exhibit chaos in classical domain and thus establish the phenomena generic to these systems.

Quantum recurrences originate from the simultaneous excitation of discrete quantum levels [6]. The existence of recurrences has been investigated in atomic [7–15] and molecular [16–19] wave packet evolution. Study of some of the periodically driven quantum systems [20–22], and two-degree-of-freedom systems such as stadium billiard [23] indicates the presence of quantum recurrences in higher dimensional systems as well. Recently it is proved that the quantum recurrences are generic to periodically driven system which may display chaos in their classical counterpart [24]. In this paper, we study the phenomena of dynamical recurrences in general higher dimensional system, and provide mathematical foundations to the phenomena by calculating recurrence times, using semiclassical secular perturbation theory. At these time scales, namely classical period and quantum recurrence time, an initial excitation produced in the system displays its full or partial reappearance.

A quantum wave packet in its early evolution follows classical mechanics and it reappears after a classical period following classical trajectory. Later, following wave mechanics it spreads and collapses, however, the discreteness of the quantum world leads to its restructuring. We show that, (i) the phenomena of dynamical recurrences occur in higher dimensional quantum systems provided that at least two of its degrees of freedom are coupled. (ii) Furthermore we explain that, (a) the nonlinearity of the uncoupled systems, and, (b) the initial conditions on the excitation contribute to the classical and the quantum recurrence times occurring in the coupled multi-dimensional systems. (iii) We also study the interdependence of these time scales for different kinds of dynamical systems.

We write the general Hamiltonian of a system with any of its two degrees of freedom coupled [25,26], as

$$H = H_0(\mathcal{I}) + \lambda H_c(\mathcal{I}, \theta) \tag{1}$$

where, H_0 is the Hamiltonian of the system in the absence of coupling, expressed in the action coordinates $\mathcal{I} = (I_1, I_2)$. Moreover, H_c is the coupling Hamiltonian which describes coupling between $\mathcal{I}$ and is periodic in angle, $\theta = (\theta_1, \theta_2)$, for nonlinear resonances in the system. The parameter λ describes the strength of the coupling. We express the coupling Hamiltonian as,

$$H_c = \sum_n H_n(\mathcal{I}) e^{in\theta}, \tag{2}$$

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where $n = (n_1, n_2)$. Whenever the frequencies $\Omega = \partial H_0/\partial \mathcal{I}$ obey the relation, $n\Omega = n_1\Omega_1 + n_2\Omega_2 = 0$, resonances occur in the system.

We may find classical chaos in the two degrees of freedom system, as a function of the coupling strength, whenever the two degrees of freedom are coupled. We consider that the coupling exists between I_1 and I_2. Within the region of resonance we find slow variations in action, hence, following the method of secular perturbation theory [27], we average over faster frequency, and get the averaged Hamiltonian for the Nth resonance, as $\bar{H} = \bar{H}_0(I) + \lambda V \cos(N\varphi)$. Here, $\varphi = \theta_1 - (M/N)\theta_2$, $I = I_1$ is the action corresponding to the angle θ_1, V is the Fourier amplitude, and M and N are relatively prime integers [25,26]. The action I_2 becomes the constant of motion. Moreover, $\bar{H}_0(I)$ expresses uncoupled averaged Hamiltonian. In case the coupling exists between $\mathcal{N}$ degrees of freedom, we may apply the semi-classical secular perturbation technique $(\mathcal{N} - 1)$ times to study the effect of most dominant degree of freedom.

The energy of the excitation changes slowly when we produce it in the vicinity of Nth resonance of the higher dimensional system and it is narrowly peaked. Therefore, we expand the unperturbed energy, $\bar{H}_0(I)$, by means of Taylor's expansion around mean action I_0, and keep only the terms up to second order [28]. As a result, we express the Hamiltonian of the system, governing the evolution of the excitation in the vicinity of Nth resonance, as

$$\bar{H} \cong \bar{H}_0(I_0) + \omega(I - I_0) + \frac{\zeta}{2}(I - I_0)^2 + \lambda V \cos(N\varphi). \quad (3)$$

Here, $\bar{H}_0(I_0)$ is the energy of the uncoupled system at the action $I = I_0$, and ω expresses the first order derivative of $\bar{H}_0$ with respect to action calculated at I_0 and defines classical frequency of the excitation. The parameter ζ defines the nonlinear dependence of the energy of the system on the quantum number.

We introduce the transformation $N\varphi = 2\theta$ and quantize the dynamics around the Nth resonance by quantizing the action [29,30], that is,

$$I - I_0 = \frac{\hbar}{i}\frac{\partial}{\partial\varphi} = \frac{N\hbar}{2i}\frac{\partial}{\partial\theta}. \quad (4)$$

As a result, the Hamiltonian of the resonant system in the quantum domain reads

$$\bar{H} = -\frac{N^2\zeta\hbar^2}{8}\frac{\partial^2}{\partial\theta^2} + \frac{\hbar}{2i}N\omega\frac{\partial}{\partial\theta} + \bar{H}_0(I_0) + \lambda V \cos 2\theta. \quad (5)$$

Hence, the quantum mechanical system evolves according to the Schrödinger equation, $\bar{H}\psi_k = \mathcal{E}_k\psi_k$. Here, ψ_k is the wavefunction of the system in the region of a resonance and is, therefore, required to fulfill periodicity condition, i.e. $\psi_k(\theta + \pi) = \psi(\theta)$. The corresponding eigenvalue $\mathcal{E}_k$ defines the eigen energy of the system. With the help of transformation, $\psi_k = \phi_k \exp(-2i\omega\theta/(N\zeta\hbar))$, we map the Schrödinger equation on the Mathieu equation,

$$\left[\frac{\partial^2}{\partial\theta^2} + a_\nu - 2q\cos 2\theta\right]\phi_\nu = 0 \ , \quad (6)$$

where,

$$a_\nu = \frac{8}{N^2\zeta\hbar^2}\left[\mathcal{E}_\nu - \bar{H}_0 + \frac{\omega^2}{2\zeta}\right] , \quad (7)$$

and

$$q = \frac{4\lambda V}{N^2\zeta\hbar^2} \ . \quad (8)$$

The π-periodic solutions of equation (6) correspond to even functions of the Mathieu equation whose corresponding eigenvalues are real [31]. These solutions are defined by Floquet states, i.e. $\phi_\nu(\theta) = e^{i\nu\theta}P_\nu(\theta)$, where $P_\nu(\theta)$ is the even order Mathieu function.

In order to obtain a π-periodic solution in φ-coordinate we require the coefficient of φ in the exponential factor to be equivalent to an even integer number, k. This condition provides the value for the index ν as

$$\nu = \frac{2k}{N} + \frac{2\omega}{N\zeta\hbar}. \quad (9)$$

Therefore, we may express the eigen-energy of the system as

$$\mathcal{E}_k = \frac{N^2\zeta\hbar^2}{8}a_{\nu(k)}(q) - \frac{\omega^2}{2\zeta} + \bar{H}_0(I_0) \ , \quad (10)$$

where, $a_\nu(q)$ is the Mathieu characteristic parameter [31].

In order to check this result we study the case of zero coupling strength, that is $\lambda = 0$. In this case, the value for Mathieu characteristic parameter becomes $a_\nu(q = 0) = \nu^2$. This reduces the quasi-energy, $\mathcal{E}_k$, to equation (3) in the absence of coupling term, that is, $\lambda = 0$, and in addition leads to define k as $(I - I_0)/\hbar$.

The initial excitation produced at $I = I_0$ observes various time scales at which it reappears completely or partially during its evolution. In order to find these time scales at which an excitation in the quantum mechanical coupled higher dimensional system recurs, we employ the eigenenergy $\mathcal{E}_k$ of the system [33,34].

These time scales, $T_\lambda^{(i)}$, are inversely proportional to the frequencies $\omega^{(i)}$, where $\omega^{(i)} = (i!)^{-1}\hbar^{(i-1)}\partial^{(i)}\mathcal{E}_k/\partial I^{(i)}$, when calculated at $I = I_0$. The index i describes the order of differentiation of the quasi energy $\mathcal{E}_k$. With the increasing values for i we have smaller frequencies indicating longer higher-order recurrence times.

The time scale, $T_\lambda^{(1)} = T_\lambda^{(cl)}$, and defines classical period of the higher dimensional coupled system and is inversely proportional to $\omega^{(1)}$. In the absence of coupling $\omega^{(1)}$ reduces to ω. The time scale, $T_\lambda^{(2)} = T_\lambda^{(Q)}$, and defines quantum mechanical recurrence time in the higher dimensional coupled systems. It has inverse proportionality with $\omega^{(2)}$. Here, we have $\omega^{(2)} = (2!)^{-1}\hbar\partial^2\mathcal{E}_k/\partial I^2|_{I=I_0}$.

On substituting the value for $\mathcal{E}_k$ from equation (10) in the definition for $\omega^{(1)}$ and $\omega^{(2)}$, we obtain the classical period as,

$$T_\lambda^{(cl)} = [1 - M_o^{(cl)}]T_0^{(cl)}, \quad (11)$$

and the quantum recurrence time for the coupled system as,

$$T_\lambda^{(Q)} = [1 - M_o^{(Q)}]T_0^{(Q)}. \quad (12)$$

Here, the time scales $T_0^{(cl)}(\equiv 2\pi\omega^{-1})$ defines classical period and $T_0^{(Q)}(\equiv 2\pi(\frac{1}{2!}\hbar\zeta)^{-1})$ defines quantum revival time in the *absence* of coupling. The time modification factors $M_o^{(cl)}$ and $M_o^{(Q)}$ are given as,

$$M_o^{(cl)} = -\frac{1}{2}\left(\frac{\lambda V\zeta}{\omega^2}\right)^2 \frac{1}{(1-\mu^2)^2}\,, \tag{13}$$

and

$$M_o^{(Q)} = \frac{1}{2}\left(\frac{\lambda V\zeta}{\omega^2}\right)^2 \frac{3+\mu^2}{(1-\mu^2)^3} \tag{14}$$

where,

$$\mu = \frac{N\hbar\zeta}{2\omega}. \tag{15}$$

Equations (11) and (12) express the classical period and the quantum revival time in the presence of coupling between two degrees of freedom as a function of the coupling strength λ, nonlinearity ζ associated with the uncoupled system and other characteristic parameters of the system. Analysis of equations (13) and (14) leads us to the conclusion that the first terms of the modification factors $M_o^{(cl)}$ and $M_o^{(Q)}$ depend quadratically on the coupling strength λ, and, on the nonlinearity ζ present in the initial uncoupled system. Whereas they are inversely proportional to the fourth power of frequency ω in both the cases. The second terms are function of μ, defined in equation (15). Hence, for the coupling strength $\lambda = 0$ we find $T_\lambda^{(cl)} = T_0^{(cl)}$ and $T_\lambda^{(Q)} = T_0^{(Q)}$.

Case a: in the absence of nonlinearity, i.e. for $\zeta = 0$, the time modification factor for the classical period $M_o^{(cl)}$ and for the quantum recurrence time $M_o^{(Q)}$ vanish, which is evident from equations (13) and (14). Thus, for linear coupled systems the quantum recurrences take place at infinite time, i.e. $T_\lambda^{(Q)} = T_0^{(Q)} = \infty$. Nevertheless, the system displays recurrences after the classical period, $T_\lambda^{(cl)} = T_0^{(cl)} = 2\pi/\omega$. Hence, in the coupled linear higher dimensional systems only classical periods exist.

Case b: for $\mu < 1$, the time modification factors $M_o^{(cl)}$ and $M_o^{(Q)}$ are related as,

$$M_o^{(Q)} = -3M_o^{(cl)} = 3\alpha, \tag{16}$$

where

$$\alpha = \frac{1}{2}\left(\frac{\lambda V\zeta}{\omega^2}\right)^2. \tag{17}$$

Thus, the classical period $T_\lambda^{(cl)}$ and the quantum recurrence time $T_\lambda^{(Q)}$ in the presence of coupling are related with the $T_0^{(cl)}$ and $T_0^{(Q)}$ of the uncoupled system as

$$\frac{3T_\lambda^{(cl)}}{4T_0^{(cl)}} + \frac{T_\lambda^{(Q)}}{4T_0^{(Q)}} = 1. \tag{18}$$

In the absence of coupling we find $T_\lambda^{(cl)} = T_0^{(cl)}$ and $T_\lambda^{(Q)} = T_0^{(Q)}$ which fulfills equation (18).

The case may be achieved for a weak nonlinearity in the dominant degree of freedom of the coupled system, that is $\zeta \ll 1$. The quantum recurrence time in coupled system $T_\lambda^{(Q)}$ and in uncoupled system $T_0^{(Q)}$, depend inversely on nonlinearity in the system, and are therefore much larger than classical periods $T_\lambda^{(cl)}$ and $T_0^{(cl)}$ in this case.

As it follows from equation (16), the quantum recurrence time $T_\lambda^{(Q)}$ reduces by $3\alpha T_0^{(Q)}$, whereas, the classical period $T_\lambda^{(cl)}$ increases by $\alpha T_0^{(cl)}$. Hence, we may conclude that the quantum dynamical recurrence time $T_\lambda^{(Q)}$ reduces much faster than the classical period $T_\lambda^{(cl)}$ in the presence of a small nonlinearity in the system.

Since, the modification factor α displays direct proportionality with the square of the nonlinearity parameter, ζ^2, in the system. In the asymptotic limit, i.e. for ζ approaching zero, we get the result $T_\lambda^{(cl)} = T_0^{(cl)}$ and $T_\lambda^{(Q)} = T_0^{(Q)} = \infty$ as discussed in *case a*.

Case c: in the presence of a relatively larger value of the nonlinearity parameter and/or highly quantum mechanical systems, we may consider $\mu > 1$. We have the time modification factors related as

$$M_o^{(Q)} = M_o^{(cl)} = -\beta, \tag{19}$$

where

$$\beta = \frac{1}{2}\left(\frac{4\lambda V}{N^2\zeta\hbar^2}\right)^2. \tag{20}$$

Thus, the classical period $T_\lambda^{(cl)}$ and the quantum recurrence time $T_\lambda^{(Q)}$ in the presence of coupling are related with the $T_0^{(cl)}$ and $T_0^{(Q)}$ of the uncoupled system as

$$\frac{T_\lambda^{(cl)}}{T_0^{(cl)}} - \frac{T_\lambda^{(Q)}}{T_0^{(Q)}} = 0. \tag{21}$$

The time modification factors for the classical period and the quantum recurrence time vanish as β reduces to zero. Therefore, equations (11) and (12) provide us the asymptotic result, that is, $T_\lambda^{(cl)} = T_0^{(cl)}$ and $T_\lambda^{(Q)} = T_0^{(Q)}$ and we find that equation (21) holds.

The parameter β, which determines the modification both in classical period and in quantum recurrence time, is inversely dependent on the fourth power of the Planck's constant $\hbar$. Hence, for highly quantum mechanical cases we note that β approaches zero and the times of recurrence remain unchanged.

The classical period is inversely proportional to the classical frequency, $\omega^{(1)}$, which is controlled by the initial excitation energy. We note that: (i) in presence of no nonlinearity (*case a*), only classical period exists as discussed above. Hence as frequency approaches zero, the classical period becomes infinity which is the case of an open system; (ii) for small value of the nonlinearity parameter in *case b*, as frequency becomes very small the classical period $T_\lambda^{(cl)}$ and quantum recurrence time $T_\lambda^{(Q)}$ [32] changes following a square law dependence on coupling strength,

λ, as we find from equations (16) and (17). However, for larger values of the frequency the times $T_\lambda^{(cl)}$ and $T_\lambda^{(Q)}$ remain unchanged; (iii) for relatively large value of the nonlinearity parameter, as in *case c*, the situation is opposite. For smaller value of frequency we find larger value for N, therefore, the modification factor reduces to zero. Hence, from equations (19) and (20), we find that $T_\lambda^{(cl)} = T_0^{(cl)}$ and $T_\lambda^{(Q)} = T_0^{(Q)}$. However, for larger values of the frequency, we note that $T_\lambda^{(cl)}$ and $T_\lambda^{(Q)}$ vary following square law behavior as a function of modulation strength, λ.

For an excitation produced at the center of a resonance the quantum evolution of the system is characterized by equation (5) under the consideration of $\theta \approx 0$. Hence, the secular Hamiltonian of the coupled system for exact resonance case is

$$\bar{H} \approx -\frac{N^2\zeta\hbar^2}{8}\frac{\partial^2}{\partial\theta^2} - 2\lambda V\theta^2 . \qquad (22)$$

Equation (22) describes the Hamiltonian of a harmonic oscillator. Hence, this provides an evidence that if the excitation originates initially from the center of a resonance it will experience recurrences after each classical period, as shown in Figure 1a. The quantum recurrence time now is $T_\lambda^{(Q)} = T_0^{(Q)} = \infty$. Thus the evolution is the same in every dynamical system in the case of exact resonance. The oscillator frequency is $N\sqrt{\zeta\lambda V}$. Thus in case the coupling strength, λ, reduces to zero the harmonic oscillator behavior disappears. The system now possesses only the classical period $T_0^{(cl)}$ and the quantum recurrence time $T_0^{(Q)}$ of the uncoupled system, we find this behavior in Figure 1b. This effect provides us information about level spacing around the center of a resonance [35] as well. For harmonic oscillator the spacing between successive levels is equal, hence we conclude that for quasienergy levels of the Floquet operator belonging to the center of resonance the level spacing is always equal.

We conclude that in higher dimensional systems a coupling results in modifying the recurrence times available for the uncoupled systems so far as the dynamics is considered close to a nonlinear resonance. The phenomena is helpful to improve the efficiency of the Recurrence Tracking Microscope [36]. The current results may help to identify [5] quantum acceleration modes (QAM) [37] and understand the dynamics associated. Moreover the suggested treatment may reveal the understanding of the dynamics of Bose Einstein Condensates in coupled systems [38].

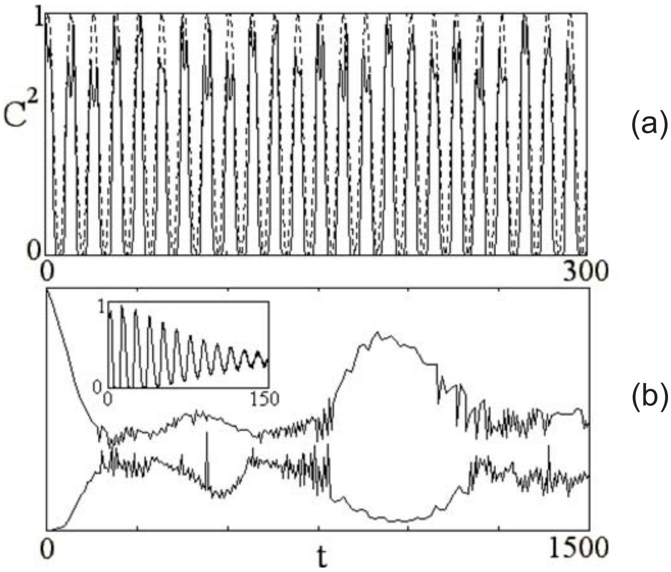

Fig. 1. The change in recurrence phenomena for the wave packet originating from the center of a resonance: (a) we plot the square of auto-correlation function, C^2, as a function of time, t, for a Gaussian wave packet. The wave packet is initially propagated from the center of a resonance in atomic Fermi Accelerator [32] and its mean position and mean momentum are $z_0 = 14.5$ and $p_0 = 1.45$, respectively. We find that the wave packet experiences recurrence after each classical period. Thick line corresponds to our numerically obtained result and dashed line indicates quantum recurrence occurring for an oscillator in harmonic motion. The classical period is calculated to be 4π when the coupling strength, $\lambda = 0.3$. As the coupling strength λ becomes zero the evolution of the wave packet completely changes. (b) We plot square of the auto-correlation function for the wave packet in the absence of coupling, i.e. $\lambda = 0$. We find that now the wave packet experiences collapse after many classical periods. Later, it displays recurrence at quantum recurrence time $T_0^{(Q)}$. The inset displays the short time evolution of the wave packet comprising many classical periods for $\lambda = 0$.

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